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Directed strongly regular graphs with $\mu = \lambda$

Leif K. Jørgensen

Department of Mathematics and Computer Science, Aalborg University, Fr. Bajers Vej 7, DK-9220 Aalborg Ø, Denmark

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Abstract

Directed strongly regular graphs with $\mu = \lambda = t - 1$ and $k - 1$ divisible by μ are constructed from cyclic groups. Nonexistence of a directed strongly regular graph with $n = 32$, $k = 6$, $t = 5$, $\mu = \lambda = 1$ is proved.

Keywords: Directed strongly regular graphs; Generalized Cayley graphs

A directed strongly regular graph with parameters (n, k, μ, λ, t) is a regular directed graph on n vertices with degree k, such that every vertex is incident with t undirected edges, and the number of paths of length 2 directed from a vertex x to a vertex y is λ if there an $x \to y$ edge and μ otherwise. We use the notation $x \to y$ if there is an edge directed from x to y and possibly also an edge directed from y to x. The adjacency matrix A of a directed strongly regular graph satisfies the equations

$$A^2 = tI + \lambda A + \mu(J - I - A) \quad \text{and} \quad JA = AJ = kJ.$$

By counting the number paths of length 2, we see that the parameters are related by the equation $k(k + (\mu - \lambda)) = t + (n - 1)\mu$.

These graphs were first investigated by Duval [1].

In this paper we consider only the case where $0 < t < k$, since otherwise the graph is either a doubly regular tournament or an undirected strongly regular graph. Then in fact $t \geqslant \max\{\mu, \lambda + 1\}$, see [1].

The case $\mu = \lambda = 1$ was considered earlier by Hammersley [2].

Duval [1] proved that a directed strongly regular graph with $t = 2$ and $\mu = \lambda = 1$ exists if k is even. In the next theorem we extend this result to odd k and some parameter sets with $t - 1 = \mu = \lambda > 1$. We use a construction known as generalized Cayley graphs, see Marušič, Scapellato and Zagaglia Salvi [5].

E-mail address: leif@math.auc.dk (L.K. Jørgensen).

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Theorem 1. *For every pair of natural numbers μ and k where μ divides $k-1$ there exists a directed strongly regular graph with parameters $(n, k, \mu, \lambda=\mu, t=\mu+1)$ where $n=(k+1)(k-1)/\mu$.*

Proof. The vertices of the graph are the integers modulo n.

The edge set is

$$\{x \to y \mid x + ky \in \{1, \ldots, k\} \pmod n\}.$$

Every vertex x has k outneighbours $k(s-x)$, $s=1,\ldots,k$, as $k^2 \equiv 1 \pmod n$. Every vertex y has k inneighbours $s-ky$, $s=1,\ldots,k$.

The graph has no loops $x \to x$, as any number congruent to $x+kx$ modulo n is divisible by $k+1$.

For any two vertices x and z we can write $z \equiv x + h \pmod n$, $0 \leqslant h \leqslant k^2-1$ in exactly μ ways if $x \neq z$ and in exactly $t=\mu+1$ ways if $x=z$. For any such number h there is a unique way to write $h=qk-r$ such that $1 \leqslant q, r \leqslant k$. We therefore have paths $x \to y \to z$, where $y \equiv k(r-x)$, since $x+ky \equiv r$ and $y+kz \equiv k(r-x+z) \equiv k(kq) \equiv q$.

For any two vertices x and z , let h_1 and h_2 be numbers such that $0 \leqslant h_i \leqslant k^2-1$ and $z \equiv x + h_i \pmod n$, for $i=1,2$. Furthermore, write $h_i = q_i k - r_i$ where $1 \leqslant q_i, r_i \leqslant k$ and $y_i \equiv k(r_i - x)$, for $i=1,2$. We must show that if $h_1 \neq h_2$, then $y_1 \neq y_2$. So suppose that $y_1 = y_2$. Since $k(r_1 - x) \equiv y_1 = y_2 \equiv k(r_2 - x) \pmod n$, we have $r_1 \equiv r_2 \pmod n$, i.e., $r_1 = r_2$. This implies that $h_1 = q_1 k - r_1 \equiv q_2 k - r_2 = h_2 \pmod k$. Since $h_1 \equiv z - x \equiv h_2 \pmod n$ and n and k are relatively prime, we then have $h_1 \equiv h_2 \pmod{nk}$. Since $nk = (k^2-1)k/\mu > k^2-1$ and $0 \leqslant h_1,\ h_2 \leqslant k^2-1$, it follows that $h_1 = h_2$.

Therefore, for a fixed vertex x, we have at least μ paths of length 2 from x to each of the other $n-1$ vertices and at least t paths of length 2 from x to x. Since $(n-1)\mu + t = \mu n + 1 = k^2$, we do not have more than the required number of paths of length 2 starting at x. $\square$

We now want to find the automorphism groups of the graphs constructed in Theorem 1. Many previous constructions of directed strongly regular graphs [1,3,4] are vertex transitive, but our graphs usually have smaller groups.

Theorem 2. *Let G denote a graph constructed in Theorem* 1. *Then the maps $\alpha : i \mapsto i + (k-1)/\mu$ and $\beta : i \mapsto k - i$ generate the automorphism group of G, which is the dihedral group of order* $2(k+1)$.

If $k = 2\mu + 1$ then $n = 4\mu + 4$ and G is a Cayley graph of this group.

Proof. Clearly, α and β are permutations of the vertex set of order $k+1$ and 2, respectively.

Suppose that $x \to y$, i.e. $x + ky \in \{1, \ldots, k\} \bmod n$. Then

$$\beta(x) + k\beta(y) = k - x + k^2 - ky \equiv k + 1 - (x + ky) \in \{1, \ldots, k\} \bmod n$$

and so $\beta(x) \to \beta(y)$. For a fixed number c the map $x \mapsto x + c$ is an automorphism if and only if for all x, y,

$$x + ky \in \{1, \dots, k\} \bmod n \quad \Leftrightarrow \quad (x + c) + k(y + c) \in \{1, \dots, k\} \bmod n,$$

i.e., if $c + kc \equiv 0 \bmod n$. Thus the map $x \mapsto x + c$ is an automorphism if and only if c is a multiple of $(k-1)/\mu$.

It follows that β and α are automorphisms. Since for every i, $\beta\alpha\beta(i) = k - ((k-i) + (k-1)/\mu) = i - (k-1)/\mu = \alpha^{-1}(i)$, α and β generate the dihedral group of order $2(k+1)$.

For fixed x and i, y is a common outneighbour of x and $x + i$ if both $x + ky$ and $x + ky + i$ belong to $\{1, \dots, k\} \bmod n$. Thus x and $x + i$ have exactly $k - 1$ common outneighbours if and only if $i = \pm 1$. Any automorphism of G will therefore preserve (or reverse) the cyclic order of $0, \dots, n-1$. Thus the automorphism group is a subgroup of the dihedral group of order $2n$. Using the above characterization of which rotations are automorphisms, we see that every automorphism of G is in the group generated by α and β.

Suppose the order of this group is equal to the number of vertices in G, i.e. $(k-1)/\mu = 2$ and suppose an automorphism $\alpha^i\beta$ has a fixed point x, then $x \equiv \alpha^i\beta(x) = k - x + i(k-1)/\mu = 2\mu + 1 - x + 2i$, and so $2x \equiv 2\mu + 1 + 2i$. But since n is even an even number and an odd number cannot be congruent modulo n. Thus $\alpha^i\beta$ has no fixed point. Clearly a non-identity automorphism of the form α^i cannot have fixed points. Thus the dihedral group acts regularly on G. $\square$

The outneighbours of vertex 0 are $k, 2k, \dots k^2$. The generators for the Cayley graph are the automorphisms that map 0 to its outneighbours, i.e. $\beta, \alpha^k, \alpha^k\beta, \alpha^{2k}, \dots, \alpha^{\mu k}, \alpha^{\mu k}\beta$. By applying the group automorphism $\alpha \mapsto \alpha^k$, $\beta \mapsto \beta$ we get the generating set $\{\alpha, \alpha^2, \dots, \alpha^\mu, \beta, \alpha\beta, \dots \alpha^\mu\beta\}$. These Cayley graphs were first found by Hobart and Shaw [3].

The smallest new graphs constructed in Theorem 1 have parameters $(24, 5, 1, 1, 2)$ and $(33, 10, 3, 3, 4)$. These graphs are not vertex-transitive, but in the first case there exists another graph which is vertex-transitive.

Theorem 3. *Let S_4 be the symmetric group on the set $\{1, 2, 3, 4\}$. Then the Cayley graph of S_4 with generators $(3\ 4), (2\ 3\ 4), (1\ 2\ 4), (1\ 3\ 2\ 4), (1\ 4)(2\ 3)$ is a directed strongly regular graph with $n = 24$, $k = 5$, $\mu = \lambda = 1$, $t = 2$.*

Above we considered directed strongly regular graphs with $\mu = \lambda$ and the smallest possible value of t. We now consider graphs with the largest possible value of t.

Suppose there exists a directed strongly regular graph with $t = k - 1$, $\mu = \lambda = 1$. Duval [1] proved that the parameters of a directed strongly regular graph satisfies $(\mu - \lambda)^2 + 4(t - \mu) = d^2$ and $d \mid 2k - (\mu - \lambda)(n - 1)$, for some positive integer d. From this we get $4(t - 1) = d^2$ and $d \mid 2(t + 1)$. Then $2d$ divides $4(t + 1) = d^2 + 8$. Since d

is even, $2d$ divides d^2 and also 8, and so either $d=2,\ t=2,\ k=3,\ n=8$ (where the unique graph was constructed in Theorem 1) or $d=4,\ t=5,\ k=6,\ n=32$.

Theorem 4. *There is no directed strongly regular graph with* $n=32,\ k=6,\ t=5,\ \mu=\lambda=1$.

Proof. Suppose there exists a directed strongly regular graph G with $k=t+1,\ \mu=\lambda=1$. Let x be a vertex in G, and $z, y_0, y_1, \ldots, y_t$ be the vertices so that $x \leftrightarrow y_i$ for $i=1,\ldots,t$, and $z \to x \to y_0$.

Let H be the subgraph of G spanned by $\{y_0, \ldots, y_t\}$. Then every vertex in H has indegree exactly 1 in H, as there is $\lambda = 1$ path of length 2 from x to a vertex y_i.

(1) Suppose that y_h, y_i, y_j is a directed path from y_h to y_j. Then a directed path y_h, x, y_j cannot exist. Thus $h=0$.

We may assume that the inneighbour of y_0 in H is y_1. Since $k-t=1$ and $x \to y_0$, $y_0 y_1$ is an undirected edge. Since y_0 has indegree 1 in H, it is not adjacent to any other vertex in H, by (1).

A path of length 2 from z to x must be of the form $z \leftrightarrow y_i \leftrightarrow x$. If y_i has an outneighbour y_j then we have two paths z, x, y_j and z, y_i, y_j. Thus y_i has no outneighbour in H. We may assume that $i=2$. Let y_j be the inneighbour of y_2. Since the inneighbour of y_j cannot be y_2, it follows from (1) that its inneighbour is y_0 and $y_j = y_1$. Then y_2 does not have any further neighbour in H. Neither does y_1 as it cannot have two outneighbours in $\{y_2, \ldots, y_t\}$.

Now in the graph spanned by $\{y_3, \ldots, y_t\}$ every vertex has indegree exactly 1 and outdegree at most 1, and, by (1), there is no path of length 2. It follows that it is a union of disjoint 2-cycles, and so t is even. In particular $t \neq 5$. $\square$

Although we constructed directed strongly regular graphs from cyclic groups and as Cayley graphs of non-abelian groups, a directed strongly regular graph cannot be a Cayley graph of an abelian group. This follows from a result of Klin et al. [4].

Theorem 5. *A directed strongly regular graph cannot be a Cayley graph of an abelian group.*

Proof. The adjacency matrix of a directed strongly regular graph can be written as $A_s + A_n$, where $A_s^{\mathrm{T}} = A_s$ and $A_n + A_n^{\mathrm{T}}$ is a $\{0,1\}$ matrix. Using eigenvalue techniques it was proved in [4] that $A_s A_n \neq A_n A_s$.

Suppose that a directed strongly regular graph is a Cayley graph of a group G with generatorset $\mathscr{S}$. Let $\mathscr{S}_s = \{g \in \mathscr{S} \mid g^{-1} \in \mathscr{S}\}$ and $\mathscr{S}_n = \{g \in \mathscr{S} \mid g^{-1} \notin \mathscr{S}\}$. Then the Cayley graphs of G generated by $\mathscr{S}_s$ and $\mathscr{S}_n$ have adjacency matrices A_s and A_n, respectively. To say that $A_s A_n \neq A_n A_s$ means that there exist vertices x and y so that the number of paths of the form $x \leftrightarrow z \to y, z \not\leftarrow y$ is not equal to the number of paths of the form $x \to z \leftrightarrow y, x \not\leftarrow z$. It follows that there exists $g \in \mathscr{S}_s$ and $h \in \mathscr{S}_n$ so that $gh \neq hg$. $\square$

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